Relaxation Schemes for the Shallow Water equations

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Overview

- Realaxation Model for Conservation Laws
  - Scalar and Systems of Cons. Laws
  - Conservation Laws with Source Terms

- Shallow Water Equations - 1D
  - Relaxation Model
  - Relaxation Schemes
  - Numerical Results

- Shallow Water Equations - 2D
  - Relaxation Model and Schemes
  - Numerical Results
Relaxation Model for Scalar CL

\[ u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \]  

(1)

Relaxation system proposed by Jin & Xin 1995

\[ u_t + v_x = 0, \]
\[ v_t + c^2 u_x = -\frac{1}{\epsilon}(v - f(u)). \]  

(2)

This system can be viewed as a regularization of (1) by the wave operator

\[ u_t + f(u)_x = -\epsilon(u_{tt} - c^2 u_{xx}). \]

Applying the Champan-Enskog expansion we get

\[ u_t + f(u)_x = \epsilon \partial_x \left( \left( c^2 - f'(u)^2 \right) \partial_x u \right) + O(\epsilon^2). \]

If the \textit{subcharacteristic condition} : \(|f'(u)| < c\) holds then a rigorous convergence analysis, for 1D scalar case, can be applied yielding at the relaxation limit \(\epsilon \to 0\) the conservation law (1). (JX, 1995)
Relaxation Model with Source term, I

For a conservation law with a source term

$$u_t + f(u)_x = q(u), \quad x \in \mathbb{R}, \ t > 0,$$
$$u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

a relaxation system considered takes the form

$$u_t + v_x = q(u),$$
$$v_t + c^2 u_x = -\frac{1}{\epsilon}(v - f(u)),$$

yielding the following regularization of (3),

$$u_t + f(u)_x = q(u) + \epsilon q(u)_t - \epsilon (u_{tt} - c^2 u_{xx}).$$

Remarks
1) Same subcharacteristic condition
2) Extra term : $\epsilon q(u)_t$.
3) In general (4) does not preserve the steady states.
4) The time discretization of (4)
Relaxation Model with Source term, II

\[
\begin{align*}
\frac{U^{n+1} - U^n}{\Delta t} + V^n_x &= q(U^{n+1}), \\
\frac{V^{n+1} - V^n}{\Delta t} + c^2 U^n_x &= -\frac{1}{\epsilon}(V^{n+1} - f(U^{n+1})),
\end{align*}
\]

is fully coupled system, not the case for the corresponding time discretization of (2). An alternative approach: we consider the following relaxation system

\[
\begin{align*}
  u_t + v_x &= 0, \\
  v_t + c^2 u_x &= -\frac{1}{\epsilon}(v - f(u)) - \frac{1}{\epsilon} R(u),
\end{align*}
\]

where \( R(u) \) is an antiderivative of \( q(u) \),

\[
R(u(x)) = \int_x^x q(u(s)) \, ds.
\]
In this case (6) provides exactly a wave-type regularization of (3),

\[ u_t + f(u)_x = q(u) - \epsilon(u_{tt} - c^2u_{xx}). \]  

(7)

Also an implicit-explicit time discretization is now possible when we treat the source terms implicitly:

\[
\frac{U^{n+1} - U^n}{\Delta t} + V^n_x = 0,
\]

\[
\frac{V^{n+1} - V^n}{\Delta t} + c^2U^n_x = -\frac{1}{\epsilon}(V^{n+1} - f(U^{n+1})) - \frac{1}{\epsilon}R(U^{n+1}).
\]

(8)

If \(|f'(u)| < c\) from (7) we recover formally (3)

System (6) preserves steady states

Initial, Boundary Cond. : \(v_0 = f(u_0), \ v_b = f(u_b)\)
**System of CL**

\[
\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = 0, \quad x \in \mathbb{R}^d, \quad u = u(x, t) \in \mathbb{R}^n, \quad t > 0
\]

\[
u(\cdot, 0) = u_0(\cdot)
\]

Relaxation model

\[
\partial_t u + \sum_{j=1}^{d} \partial_{x_j} v_j = 0,
\]

\[
\partial_t v_i + A_i \partial_{x_i} u = -\frac{1}{\epsilon} (v_i - F_i(u)), \quad i = 1, \ldots, d
\]

It's a regularization by a wave operator of order \( \epsilon \), and \( A_i \) are symmetric positive definite matrices with constant coefficients that are selected to satisfy the corresponding *sub-characteristic conditions*. 
Shallow water eqns (1D)

\[ h_t + (hu)_x = 0, \]
\[ (hu)_t + (hu^2 + \frac{g}{2}h^2)_x = -ghZ', \]

General steady states:

\[ Q = hu = \text{Cnst} \]
\[ \frac{u^2}{2} + g(h + Z) = \text{Cnst} \]

SWE is a hyperbolic system with source term
SW Relaxation Models I

Relaxation Model A

\[
\begin{align*}
    h_t + v_x &= 0 \\
    Q_t + w_x &= -ghZ' \\
    v_t + c_1^2 h_x &= -\frac{1}{\epsilon}(v - Q) \\
    w_t + c_2^2 Q_x &= -\frac{1}{\epsilon}(w - \left(\frac{Q^2}{h} + \frac{g}{2}h^2\right)) \\
\end{align*}
\]

Relaxation Model B

\[
\begin{align*}
    h_t + v_x &= 0 \\
    Q_t + w_x &= 0 \\
    v_t + c_1^2 h_x &= -\frac{1}{\epsilon}(v - Q) \\
    w_t + c_2^2 Q_x &= -\frac{1}{\epsilon}(w - \left(\frac{Q^2}{h} + \frac{g}{2}h^2\right)) + \frac{1}{\epsilon}R(Z; h) \\
\end{align*}
\]
$R(Z; h)(x) = \int_x^x g(hZ')(y)dy$

- $c_1, c_2$ are chosen according to sub-characteristic condition:

$$|\lambda_i(F')| < c_i, \quad i = 1, 2, \quad F' = \text{Jacobian of flux vector}$$

- For $Z \equiv 0$, $(A) \equiv (B)$

- For $\epsilon \to 0$ we recover the original SW system

- Both relaxation systems have linear principal part

- Implicit-explicit time discretizations for (B)

- System (B) have same steady states as the continuous problem
We consider the *Relaxation Model B* and let 

\[ u = \begin{bmatrix} h \\ Q \end{bmatrix}, \quad v = \begin{bmatrix} v \\ w \end{bmatrix}, \]

our system can be rewritten as

\[
\begin{align*}
\frac{du}{dt} + \frac{dv}{dx} &= 0, \\
\frac{dv}{dt} + C^2 \frac{du}{dx} &= -\frac{1}{\epsilon}(v - F(u)) - \frac{1}{\epsilon}S(u),
\end{align*}
\]

\[
F(u) = (Q, \frac{Q^2}{h} + \frac{g}{2} h^2)^T, \quad S(u) = (0, -\int x g(h(y)Z'(y))dy)^T
\]

where \( u, v \in \mathbb{R}^2 \) and \( C^2 \in \mathbb{R}^{2 \times 2} \) is a positive matrix.
Upwind Scheme I

We assume a uniform spaced grid with $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and a uniform time step $\Delta t = t^{n+1} - t^n$, $n = 0, 1, 2, \ldots$.

\[ u_i^n \sim \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t^n) \, dx, \quad u_i^{n+\frac{1}{2}} \sim u(x_{i+\frac{1}{2}}, t^n) \]

We start by considering the following one-step conservative system for the homogeneous case (no source term present)

\[ \frac{\partial}{\partial t} u_i + \frac{1}{\Delta x} (v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}) = 0, \]
\[ \frac{\partial}{\partial t} v_i + \frac{1}{\Delta x} C^2 (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) = -\frac{1}{\epsilon} (v_i - F(u_i)). \]

The linear hyperbolic part has two Riemann invariants (characteristic speeds) $v \pm Cu$ associated with the characteristic fields $\pm C$ respectively. The first order upwind approximation of $v \pm Cu$ is

\[ (v + Cu)_{i+\frac{1}{2}} = (v + Cu)_i, \quad (v - Cu)_{i+\frac{1}{2}} = (v - Cu)_{i+1}. \]
Upwind Scheme II

Hence,

\[ u_{i+\frac{1}{2}} = \frac{1}{2}(u_i + u_{i+1}) - \frac{1}{2}C^{-1}(v_{i+1} - v_i), \]

\[ v_{i+\frac{1}{2}} = \frac{1}{2}(v_i + v_{i+1}) - \frac{1}{2}C(u_{i+1} - u_i). \]

First order upwind semi-discrete approximation of the relaxation scheme:

\[ \frac{\partial}{\partial t} u_i + \frac{1}{2\Delta x} (v_{i+1} - v_{i-1}) - \frac{1}{2\Delta x} C(u_{i+1} - 2u_i + u_{i-1}) = 0, \]

\[ \frac{\partial}{\partial t} v_i + \frac{1}{2\Delta x} C^2(u_{i+1} - u_{i-1}) - \frac{1}{2\Delta x} C(v_{i+1} - 2v_i + v_{i-1}) = -\frac{1}{\epsilon}(v_i - F(u_i)) \]

\[ -\frac{1}{\epsilon}S(u_i), \]

where

\[ S(u_i) = \begin{bmatrix} 0 \\ -\int_{x_i}^{x_{i+1}} gh(y)Z'(y)dy \end{bmatrix}. \]
MUSCL Scheme, I

We replace the piecewise constant approximation by a MUSCL piecewise linear interpolation: for the $k$-th component of $v \pm Cu$ we have:

\[
(v + c_k u)_{i + \frac{1}{2}} = (v + c_k u)_i + \frac{1}{2} \Delta x s^+_i, \\
(v - c_k u)_{i + \frac{1}{2}} = (v - c_k u)_{i + 1} - \frac{1}{2} \Delta x s^-_{i + 1},
\]

where $u, v$ are the $k$-th components of $v, u$ and the slopes $s^\pm$ in the $i$-th cell:

\[
s^\pm_i = \frac{1}{\Delta x} (v_{i + 1} \pm c_k u_{i + 1} - v_i \mp c_k u_i) \phi(\theta^\pm_i), \\
\theta^\pm_i = \frac{v_i \pm c_k u_i - v_{i - 1} \mp c_k u_{i - 1}}{v_{i + 1} \pm c_k u_{i + 1} - v_i \mp c_k u_i},
\]

where $\phi$ is a limiter function satisfying $0 \leq \phi(\theta) \leq \text{minmod}(2, 2\theta)$.

- MinMod (MM): $\phi(\theta) = \max(0, \min(1, \theta))$
- VanLeer (VL): $\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}$
- Monotonized Central (MC): $\phi(\theta) = \max(0, \min((1 + \theta)/2, 2, 2\theta))$
MUSCL Scheme, II

Second order semi-discrete relaxation scheme (componentwise form)

\[
\frac{\partial}{\partial t} u_i + \frac{1}{2\Delta x}(v_{i+1} - v_{i-1}) - \frac{c_k}{2\Delta x}(u_{i+1} - 2u_i + u_{i-1}) \\
- \frac{1}{4}(s_{i+1}^- - s_i^- + s_{i-1}^+ - s_i^+) = 0,
\]

\[
\frac{\partial}{\partial t} v_i + \frac{c_k^2}{2\Delta x}(u_{i+1} - u_{i-1}) - \frac{c_k}{2\Delta x}(v_{i+1} - 2v_i + v_{i-1}) \\
+ \frac{c_k}{4}(s_{i+1}^- - s_i^- - s_{i-1}^+ + s_i^+) = -\frac{1}{\varepsilon}(v_i - F_k(u_i)) - \frac{1}{\varepsilon}S_k(u_i),
\]

with $S_k, F_k$ being the $k$–th components of $S, F$ respectively.
**Fully Discrete Schemes, I**

A first order in time RK-type scheme, \((Z \equiv 0)\)

(A) Given \(u^n, v^n\) apply a finite volume method to update \(u, v\) over time \(\Delta t\) by solving the homogeneous linear hyperbolic system

\[
\begin{bmatrix}
    u \\
    v
\end{bmatrix}_t + \begin{bmatrix}
    0 & I \\
    C^2 & 0
\end{bmatrix}
\begin{bmatrix}
    u \\
    v
\end{bmatrix}_x = \begin{bmatrix}
    0 \\
    0
\end{bmatrix},
\]

and obtain new values \(u^{(1)}, v^{(1)}\).

(B) Update \(u^{(1)}, v^{(1)}\) to \(u^{n+1}, v^{n+1}\) by solving the equations,

\[
\begin{align*}
    u_t &= 0, \\
    v_t &= -\frac{1}{\epsilon}(v - F(u)),
\end{align*}
\]

over time \(\Delta t\).
A second order in time RK-type scheme,

\[
\begin{align*}
    u^{n,1} &= u^n, & v^{n,1} &= v^n + \frac{\Delta t}{\epsilon}(v^{n,1} - F(u^{n,1})) + \frac{\Delta t}{\epsilon}S(u^{n,1}); \\
    u^{(1)} &= u^{n,1} - \Delta tD_+v^{n,1}, & v^{(1)} &= v^{n,1} - \Delta tC^2D_+u^{n,1}; \\
    u^{n,2} &= u^{(1)}, \\
    v^{n,2} &= v^{(1)} - \frac{\Delta t}{\epsilon}(v^{n,2} - F(u^{n,2})) - \frac{2\Delta t}{\epsilon}(v^{n,1} - F(u^{n,1})) \\
    &\quad - \frac{\Delta t}{\epsilon}S(u^{n,2}) - \frac{2\Delta t}{\epsilon}S(u^{n,1}); \\
    u^{(2)} &= u^{n,2} - \Delta tD_+v^{n,2}, & v^{(2)} &= v^{n,2} - \Delta tC^2D_+u^{n,2}; \\
    u^{n+1} &= \frac{1}{2}(u^n + u^{(2)}), & v^{n+1} &= \frac{1}{2}(v^n + v^{(2)}).
\end{align*}
\]

where

\[
D_+w_i = \frac{1}{\Delta x}(w_{i+\frac{1}{2}} - w_{i-\frac{1}{2}}).
\]
Choice of parameters

■ CFL Condition

1st order scheme : \(\max\{c_1, c_2\} \frac{\Delta t}{\Delta x} \leq 1\)

2nd order scheme : \(\max\{c_1, c_2\} \frac{\Delta t}{\Delta x} \leq \frac{1}{2}\)

■ Choice of \(c_1, c_2\) : based on rough estimates of the eigenvalues : \(u \pm \sqrt{gh}\) and satisfy the subcharacteristic condition

\[
c_1 \geq \sup |u + \sqrt{gh}| \quad \text{and} \quad c_2 \geq \sup |u - \sqrt{gh}|
\]

\[
c_1 = c_2 = \max \left\{ \sup |u + \sqrt{gh}|, \sup |u - \sqrt{gh}| \right\}
\]

■ Choisce of \(\epsilon\) : \(\epsilon << \Delta x, \epsilon << \Delta t\)
Dam Break Flow $Z \equiv 0$

We consider a channel of length $L = 2000m$. A dam is located at $x_0 = 1000m$ and at time $t = 0$ the dam collapses. We compute the solution for time $T = 50s$ with initial conditions:

$$u(x, 0) = 0, \quad h(x, 0) = \begin{cases} h_1 & x \leq 1000, \\ h_0 & x > 1000, \end{cases}$$

with $h_1 > h_0$. This is the Riemann problem for the homogeneous problem. The flow consists of a shock wave (bore) travelling downstream and a rarefaction wave (depression wave) travelling upstream. The upstream depth $h_1$ was kept constant at 10m, while the downstream depth $h_0$ was different for each problem.

- $h_0/h_1 > 0.5$ : subcritical flow
- $h_0/h_1 < 0.5$ : subcritical upstream, supercritical downstream
- $h_0/h_1 << 0.5$ : strongly supercritical downstream
- $CFL = 0.5m$, $\Delta x = 20m$, $c_1 = 5$, $c_2 = 12$, $\epsilon = 1.E - 4$
Figure 1: Dam-break flow, $h_0/h_1=0.5$, (x) Upwind and (o) MUSCL with MC limiter.
Figure 2: Dam-break flow, $h_0/h_1=0.05$, (x)Upwind and (o)MUSCL with MC limiter.
Figure 3: Dam-break flow, $h_0/h_1=0.005$, ($\times$)Upwind and (o)MUSCL with MC limiter.
Dry Bed problem, $h_0 = 0$

This a challenging problem as a result of the singularity that occurs at the transition point of the advancing front. We compute the solution at $T = 40s$

- No modifications to the scheme to incorporate the dry area
- Globally accurate results free of oscillations
- The water height and discharge remain positive
- The transition point between the wet and the dry zone is close to the exact one, but some difficulties appear on the velocity.
- Overall the solution is stable, monotone with no special front tracking techniques
- $CFL = 0.5 \Delta x = 10m, c_1 = 18, c_2 = 16, \epsilon = 1.E^{-4}$
Dry Bed problem, Water Height

Figure 4: Dry bed dam-break flow \((h)\), (o)MUSCL with MM limiter.
Figure 5: Dry bed dam-break flow ($q$), (o)MUSCL with MM limiter.
Dry Bed problem, Velocity

Figure 6: Dry bed dam-break flow ($u$), (o)MUSCL with MM limiter.
**Flow at Rest,** \( Z \neq 0 \)

We consider a channel of length \( L = 25m \) with a non-trivial bathymetry \( Z \), with initial conditions

\[
\begin{align*}
  u(x, 0) &= 0, \quad \forall x \in \mathbb{R}, \\
  h(x, 0) + Z(x) &= H, \quad \forall x \in \mathbb{R},
\end{align*}
\]

**Exact solution**

\[
\begin{align*}
  u(x, t) &= 0, \quad \forall x \in \mathbb{R}, t \geq 0, \\
  h(x, t) + Z(x) &= H, \quad \forall x \in \mathbb{R}, t \geq 0,
\end{align*}
\]

\[
Z(x) = \begin{cases} 
  0.2 - 0.05(x - 10)^2, & 8 \leq x \leq 12, \\
  0, & \text{otherwise},
\end{cases}
\]

with \( H = 2m, \quad \epsilon = 1.E - 5, \quad c_1 = 4, c_2 = 4.5, \quad CFL = 0.5, \quad T = 200s, \quad \Delta x = 0.125 \)
Flow at Rest, Water Height

Figure 7: Flow at rest (water height): (+) standard source, (o) integral source
Figure 8: Flow at rest (discharge): (+) standard source, (o) integral source
Figure 9: Flow at rest: Magnified view of the discharge.
\( \epsilon \)-dependance of the solution

The variance of the values of the water level as well as of the discharge from the steady states, as \( \epsilon \to 0 \) are of \( O(\epsilon) \) as can be seen in Table.

Table 1: \( \ell_1 \) errors for water level and discharge (\( CFL = 0.5 \)).

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>( \ell_1 ) error for ( h )</th>
<th>Rate(( h ))</th>
<th>( \ell_1 ) error for ( q )</th>
<th>Rate(( q ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1.E-1 )</td>
<td>( 3.212E-3 )</td>
<td>( - )</td>
<td>( 2.327E-2 )</td>
<td>( - )</td>
</tr>
<tr>
<td>( 8.E-2 )</td>
<td>( 2.271E-3 )</td>
<td>( 1.551 )</td>
<td>( 1.921E-2 )</td>
<td>( 0.860 )</td>
</tr>
<tr>
<td>( 6.E-2 )</td>
<td>( 1.425E-3 )</td>
<td>( 1.619 )</td>
<td>( 1.478E-2 )</td>
<td>( 0.908 )</td>
</tr>
<tr>
<td>( 4.E-2 )</td>
<td>( 7.383E-4 )</td>
<td>( 1.628 )</td>
<td>( 1.000E-2 )</td>
<td>( 0.954 )</td>
</tr>
<tr>
<td>( 2.E-2 )</td>
<td>( 2.646E-4 )</td>
<td>( 1.480 )</td>
<td>( 5.154E-3 )</td>
<td>( 0.962 )</td>
</tr>
</tbody>
</table>
Non trivial steady states

We consider with the convergence towards steady flow over the parabolic hump in a channel of length $L = 25m$.

Depending on the boundary conditions the flow maybe subcritical, transcritical with a shock or without a shock. In all cases we use MUSCL scheme with

\[
CFL = 0.5, \quad \Delta x = 0.125m, \quad T = 200s, \quad \epsilon = 1.E - 5, \quad c_1 = 5, \quad c_2 = 7
\]

\[
u(x, 0) = 0, \quad \forall x \in \mathbb{R},
\]

\[
h(x, 0) + Z(x) = H_0, \quad \forall x \in \mathbb{R},
\]

where $H_0$ water level downstream.

- **Subcritical Flow**: $q_{up} = 4.42m^2/s, \quad H_0 = 2m$

- **Transcritical Flow without shock**: $q_{up} = 1.53m^2/s, \quad H_0 = 0.66m$

- **Transcritical Flow with shock**: $q_{up} = 0.18m^2/s, \quad H_0 = 0.33m$
Figure 10: Subcritical flow over a hump ($h$).
Subcritical Flow: Discharge

Figure 11: Subcritical flow over a hump ($q$).
Figure 12: Transcritical flow over a hump ($h$).
Figure 13: Transcritical flow over a hump ($q$).
Figure 14: Transcritical flow with shock ($h$)
Transcritical Flow with shock: Discharge

Figure 15: Transcritical flow with shock ($q$)
Drain on a non-flat bottom.

- Difficult problem since it involves the calculation of dry areas.

- BC’s: Upstream reflective, Downstream dry bed

- IC’s: \( h + Z = 0.5m \) and \( q = 0m^3/s \)

- Solution: a state at rest, on the left part of the hump with \( h + Z = 0.2m \) with \( q = 0m^3/s \) and a dry state (i.e. \( h = 0 \) and \( q = 0m^3/s \)) on the right of the hump.

- MUSCL scheme with \( \Delta x = 0.1m, \ CFL = 0.5, \, \epsilon = 1\times10^{-6}, \, c_1 = c_2 = 3.5 \)

- No modification of the method to overcome the dry area problem of zero depth and discharge.
Drain on a non-flat bottom: Water Level

Figure 16: Drain on a non-flat bottom ($h$)
Drain on a non-flat bottom: Discharge

Figure 17: Drain on a non-flat bottom ($q$)
The 2D Shallow Water Equations

\[ U_t + F(U)_x + G(U)_y = S(U); \quad (x, y) \in \Omega, \quad t \geq 0 \]

\[
U = \begin{pmatrix} h \\ hu_1 \\ hu_2 \end{pmatrix} = \begin{pmatrix} h \\ q_1 \\ q_2 \end{pmatrix}, \quad S(U) = \begin{pmatrix} 0 \\ -gh \frac{\partial Z}{\partial x}(x, y) - gh S^x_f \\ -gh \frac{\partial Z}{\partial y}(x, y) - gh S^y_f \end{pmatrix},
\]

\[
F(U) = \begin{pmatrix} q_1 \\ \frac{q_1^2}{h} + \frac{1}{2} gh^2 \\ \frac{q_1 q_2}{h} \end{pmatrix}, \quad G(U) = \begin{pmatrix} q_2 \\ \frac{q_1 q_2}{h} \\ \frac{q_2^2}{h} + \frac{1}{2} gh^2 \end{pmatrix}.
\]

\[ S^x_f = n_m^2 u_1 \sqrt{u_1^2 + u_2^2 h^{-4/3}} \]

\[ S^y_f = n_m^2 u_2 \sqrt{u_1^2 + u_2^2 h^{-4/3}}, \]

where \( n_m \) is the Manning roughness coefficient.
Relaxation System for 2D SWE

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_t + \begin{bmatrix}
  0 & I & 0 \\
  C^2 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_x + \begin{bmatrix}
  0 & 0 & I \\
  0 & 0 & 0 \\
  D^2 & 0 & 0
\end{bmatrix}\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}_y = \begin{bmatrix}
  0 \\
  -\frac{1}{\epsilon}(v - F(u) + \tilde{S}(u)) \\
  -\frac{1}{\epsilon}(w - G(u) + \tilde{\tilde{S}}(u))
\end{bmatrix}
\]

\[\tilde{S}(u) = \begin{bmatrix}
  0 \\
  -\frac{1}{2} \int_x^x gh(s, y) \frac{\partial Z}{\partial x}(s, y) ds \\
  -\frac{1}{2} \int_x^x gh(s, y) \frac{\partial Z}{\partial y}(s, y) ds
\end{bmatrix}\]

\[\tilde{\tilde{S}}(u) = \begin{bmatrix}
  0 \\
  -\frac{1}{2} \int_y^y gh(x, s) \frac{\partial Z}{\partial x}(x, s) ds \\
  -\frac{1}{2} \int_y^y gh(x, s) \frac{\partial Z}{\partial y}(x, s) ds
\end{bmatrix}\]

Subcharacteristic condition:

\[\frac{\lambda_i^2}{c_i^2} + \frac{\mu_i^2}{d_i^2} \leq 1, \quad \forall \ i = 1, 2, 3,\]

with \(\lambda_i, \mu_i\) the eigenvalues of \(\partial F(u)/\partial u\) and \(\partial G(u)/\partial u\) respectively.
Fully Discrete Schemes

- Upwind scheme: 1st order in space and time
- MUSCL scheme: 2nd order in space and time
- CFL condition, guarantees the TVD property of both schemes

\[
CFL = \max \left( \left( \max_i c_i \right) \frac{\Delta t}{\Delta x}, \left( \max_i d_i \right) \frac{\Delta t}{\Delta y} \right) \leq \frac{1}{2}.
\]

- Initial, Boundary Cond.: \( v_0 = F(u_0), \ w_0 = G(u_0), \ v_b = F(u_b), \ w_b = G(u_b) \)

- Choice of \( c_k, d_k, k = 1, 2, 3 \):
  
  1. rough estimates of the eigenvalues \( (u_1, u_1 \pm \sqrt{gh}) \) and \( (u_2, u_2 \pm \sqrt{gh}) \)
  2. calculate \( c \) and \( d \) locally at every cell as
     
     \[
     c_{i+\frac{1}{2},j} = \max_{u \in \{u_{i+\frac{1}{2},j}, u_{i-\frac{1}{2},j}\}} \left| \partial F(u)/\partial u_k \right|
     \]
     \[
     d_{i,j+\frac{1}{2}} = \max_{u \in \{u_{i,j+\frac{1}{2}}, u_{i,j-\frac{1}{2}}\}} \left| \partial G(u)/\partial u_k \right|
     \]
  3. global choice: \( c_k = d_k = \max_{i,j} (c_{i+\frac{1}{2},j}, d_{i,j+\frac{1}{2}}) \)

- \( \Delta t \gg \epsilon \) and \( \Delta y, \Delta x \gg \epsilon \)
The dam, located in the center of a channel breaks instantaneously.

- No friction ($n_m = 0$). $h_u = 10m$ and $h_d = 5, 0.1, 0m$

- Channel: $200m \times 200m$, $41 \times 41$ square grid.

- The breach is $75m$ in length, $30m$ from the left bank, $95m$ from the right.

- BC’s: $x = 0$ and $x = 200m$ transmissive and all other boundaries are reflective.

- 2nd order MUSCL scheme

- $\epsilon = 10^{-6}$ and $c_1 = 10, c_2 = 6, c_3 = 11, d_1 = 10, d_2 = 5, d_3 = 11$

- $T = 7.2s$
Figure 18: at $T = 7.2s$, Water depth, depth contours, velocity field
2D Partial Dam-Break, $h_d = 0.1m$

Figure 19: at $T = 7.2s$, Water depth, depth contours, velocity field
2D Partial Dam-Break, $h_d = 0\text{m}$

Figure 20: at $T = 7.2s$, Water depth, depth contours, velocity field
2D Partial Dam-Break - Movie
Circular Dam Break

- A two dimensional Riemann problem for the 2D SWEs

- Two regions of still water separated by a cylindrical wall with radius 11m centered in a channel. The water depth within the cylinder is 10m and 1m outside.

- The wall is removed instantaneously, the bore waves will spread and propagate radially and symmetrically

- There is a transition from subcritical to supercritical flow.

- 2nd order MUSCL scheme

- Channel: 50 x 50m, 51 x 51 square grid

- $\epsilon = 10^{-6}$ and $c_1 = c_3 = 12$, $c_2 = 7$, $d_1 = d_3 = 12$, $d_2 = 7$, $T = 0.69s$
Figure 21: at $t = 0.69s$, Water depth, depth contours, velocity field, (MM, VL limiter)
Figure 22: at $t = 0.69s$, Water depth, depth contours, velocity field, (MC, SB limiters)
Circular Dam Break, Dry Bed

Figure 23: at $t = 0.69s$, Water depth, depth contours, velocity field, (VL limiter)
Consider a channel 75\(m\) long and 30\(m\) wide

A dam is situated at \(x = 16\)m with initial water depth \(h + Z = 1.875\)m while the rest of the channel is considered dry.

The topography consists of three mounds located in the channel bottom.

Manning coefficient \(n_m = 0.018, c_i = d_i = 5, \epsilon = 1.E - 8\)
Dam Break in channel with topography, movie
Conclusions

- Relaxation Schemes for SW which combine
  - Simplicity
  - Robustness
  - Efficiency
  - Riemann solver free

- Novel ways to incorporate source terms

- Small errors of order of $\epsilon$ while preserving steady states.

- The benchmark tests show that the schemes provide accurate solutions in good agreement with well known analytical solutions.

- Comparable solutions with well known solvers

- Can be considered for practical applications?
References


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